Partition of Networks into Basins of Attraction

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We study partition of networks into basins of attraction based on a steepest ascent search for the node of highest degree. Each node is associated with, or "attracted" to its neighbor of maximal degree, as long as the degree is increasing. A node that has no neighbors of higher degree is a peak, attracting all the nodes in its basin. Maximally random scale-free networks exhibit different behavior based on their degree distribution exponent γ: for small γ (broad distribution) networks are dominated by a giant basin, whereas for large γ (narrow distribution) there are numerous basins, with peaks attracting mainly their nearest neighbors. We derive expressions for the first two moments of the number of basins. We also obtain the complete distribution of basin sizes for a class of hierarchical deterministic scale-free networks that resemble random nets. Finally, we generalize the problem to regular networks and lattices where all degrees are equal, and thus the attractiveness of a node must be determined by an assigned weight, rather than the degree. We derive the complete distribution of basins of attraction resulting from randomly assigned weights in one-dimensional chains.

I. INTRODUCTION

Networks often have heterogenous structure, with different nodes highly varying in their connectivity and in their roles [1, 2, 3, 4]. The problem of identifying these roles and assigning nodes to communities or modules based on their function is of great interest, with many methods and algorithms recently proposed [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. These methods aim to incorporate knowledge of the global network's structure with information of the nodes' local connections, to generate a network partition. However, on a more fundamental level, nodes can be simply distinguished according to the node that is their "authority", "attractor", or, in heterogeneous networks, their "hub".

The hub that each node belongs to is found by moving recursively onto the neighbor of highest degree, or number of connections, until the hub is reached — a node whose degree is greater than that of all of its neighbors. Classifying nodes by their hubs leads to a natural partition of the network into basins of attraction. See Fig. 1 for a schematic illustration. This partitioning provides a quick and easy way to classify nodes based on their relation with the network's major players, without resorting to external information.

In general, when each node is associated with a value of a scalar field, a "gradient network" emerges by replacing all the links that emanate from a node by a single directed link that points to the node's neighbor with the highest value of the field [18, 19]. Thus, recursively following nodes of highest degree is equivalent to traversing the "gradient network" formed by considering the scalar field defined by the degrees of the nodes. Many properties of gradient networks have been studied, such as the emerging degree distribution and its relation to the original network topology, and the possibility of congestion when too few nodes are receiving the flow that is generated by the gradient [18, 19]. Gradient networks have also proved useful in the analysis of energy landscapes [20], and as the basis for new and improved synchronization [21] and routing [22] methods. Here, we focus on the specific case where the value associated with each node is the degree, and thus does not require any external information but the bare topology. The walk up the degree gradient identifies each node with one of the network hubs.

The decomposition into steepest-ascent basins is of interest in many systems. For example, suggested routing schemes in communication networks involve transmitting all packets through the hub nearest to the source [23, 24]. The size of the basins delimits the performance of such routing algorithm. In a different field, an analysis of the energy landscape's network of atomic clusters shows that the energy of a configuration, or a node, decreases with the number of configurations kinetically connected to it, which is its degree [25]. Thus, as the system is cooled and its energy decreases, configurations with higher degrees tend to be visited. The actual partitioning into basins determines roughly whether the system would inevitably end up in the "ideal glass state" or arrive at one of many meta-stable states, depending on the initial conditions [26, 27].

The topology of the basins is also important if one is interested in a local strategy for finding the most connected node. A network with a single basin would make
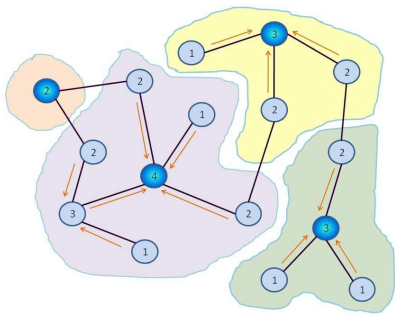
FIG. 1: (Color online) Decomposition of a network into basins of attraction. The number inside each node indicates its degree, and the arrows point from a node to its attractor — the neighbor with highest degree. Nodes with no neighbors of higher degree are *peaks* and are highlighted in the schematic. A peak represents a basin of attraction: all nodes which are attracted to it belong to its basin of attraction. The basins in this schematic have different background colors.

a steepest ascent search (in the "degree space") successful, while a more complicated topology would require a more sophisticated approach. Finally, the properties of the basins of attraction can be used to classify networks with similar degree distributions but otherwise different topology and function.

The algorithmic aspects of the partition method are relatively simple and will be discussed briefly below. Our main goal is to study, analytically and numerically, the statistical properties of the basins of attraction in ensembles of maximally random scale-free (SF) networks [1, 2, 3, 4]. We find that the topology of basins (i.e. their number, sizes, hubs' degree, etc.) shows a strong dependence on the degree distribution, and we quantify this behavior. We then study the basins' topology in a class of deterministic hierarchical SF networks [28, 29] and show that it reflects some prominent properties of the random networks. Finally, we generalize the problem to the case where the "attractiveness" of each node is determined by a random number, a 'height' rather than its degree, and derive analytical results for the basins of attraction in regular one- and two-dimensional lattices.

II. DEFINITIONS

We focus on (undirected) SF networks, i.e., networks in which node degree is broadly distributed, usually in the form $\mathcal{P}(k) \sim k^{-\gamma}$ $(m \leq k \leq N)$, where k is the degree, m is the minimum possible degree, N is the total number of nodes, and $\gamma > 2$ is the degree exponent. Many real world networks were shown to be scale-free with $\gamma < 3$ [1, 2, 3, 4]. We nevertheless study networks with $\gamma \in [2, 5]$, so as to reach the near-homogenous limit where the degree is narrowly distributed. Our networks are static and maximally random, generated according to the configuration model [30]: we first draw nodes' degrees

based on the prescribed distribution, then randomly connect open links until all nodes have all of their links connected.

A precise definition of basins of attraction requires dealing with several ambiguities (e.g., how to resolve tie-breaks). We opt for the following rules:

1. Start the search from node i with degree k_i and neighbors $j_1, j_2, ..., j_{k_i}$.

2. Denote the neighbor that has the highest degree as $j_{\max}$, with degree $k_{j_{\max}} = \max\{k_{j_1}, k_{j_2}, ..., k_{j_{k_i}}\}$. If the highest degree is shared by more than one neighbor, choose one of them arbitrarily.

3. If $k_i < k_{j_{\max}}$, i is attracted to $j_{\max}$ and both belong to the same basin of attraction.

4. If $k_i \geq k_{j_{\max}}$, node i is a *peak*, and is attracted to itself, forming a basin of attraction with all nodes (if exist) that are attracted to it.

5. Repeat for all unassigned nodes as the root of the search. Each node now belongs to exactly one basin of attraction.

Note that we require k_i to be strictly smaller than $k_{j_{\max}}$ for i to be attracted to $j_{\max}$; in other words, a node may be a peak even if it has neighbors with equal degree. This choice saves us from delving into further subtleties. The results are qualitatively the same independent on details of the definition (see Appendix A for a short discussion).

A simple and fast partitioning algorithm relies on scanning the nodes in descending degree order. Then, each node is either designated as a peak, or assigned to the basin of its neighbor with highest degree. Because we scan by degree order, we are guaranteed that the neighbor was already assigned to a basin. Thus, the running time of the algorithm (for sparse networks) is of the order of $N \log N$, the time it takes to sort the nodes [31].

III. TOPOLOGY OF BASINS OF ATTRACTION IN RANDOM SCALE-FREE NETWORKS

We attempt to capture the topology of the basins through a few representative quantities which we define below. Denote the total number of basins by N_b. Define the density of basins as the number of basins *per node* and denote it as $n_b \equiv N_b/N$. Denote next the basin size by s. The probability of a basin to be of size s is $P(s)$. A related quantity is the probability of a node to belong to a basin of size s: $Q(s) = N_b P(s)s/N = n_b s P(s)$ (and a particularly interesting case is the probability of a node to be a solitary basin $Q(1)$). Both P and Q are normalized probability distributions. Other measures of interest are the degrees of the peaks and the size of the largest basin S.

In Figs. 2-5 we present simulation results for SF networks with $N = 1000$, minimum degree $m = 1, 2$ and
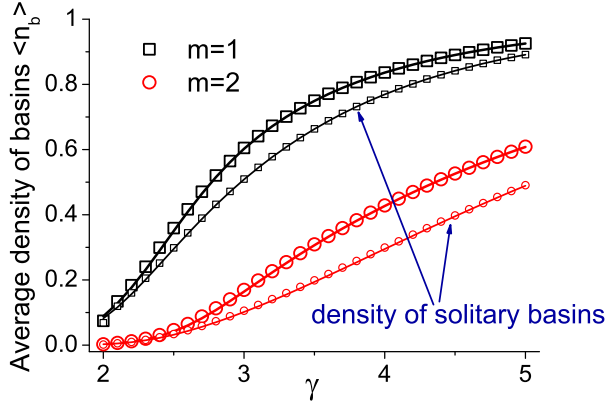
FIG. 2: Average density of basins in SF networks. Plotted is $\langle n_b \rangle$ vs. γ, as well as the density of solitary basins, $Q(1)$. Simulation results (symbols), for networks with $N = 1000$ and $m = 1, 2$, are matched by theory, Eqs. (2) and (B 2) (solid lines).

FIG. 3: Distribution of basin sizes in SF networks. Plotted is $Q(s)$ for three values of γ ($N = 1000$, $m = 1$). Inset: The basin exponent α, characterizing the power-law decay at small s, plotted vs. the degree exponent γ.

FIG. 4: Degree of the peaks in SF networks. Plotted is the distribution of the degree of the peaks for three values of γ ($N = 1000$ and $m = 1$). Inset: For large γ, the exponent β characterizing the decay of the pdf (circles) follows γ (solid line) very closely.

varying γ. The following picture emerges from the results. For small γ close to 2, the network is dominated by one hub, attracting most of the nodes to form a giant basin. Thus, the number of basins is relatively small and the size of the largest basin is narrowly distributed about $S \sim N$. The sizes of the basins and the degrees of the peaks show a bimodal distribution: a peak close to N and a fast decay for small basins which are not included in the giant basin.

For large γ, a different behavior is observed. The number of basins, N_b, is large, and most of the basins are small. The largest basin is no longer giant, and its average size scales as $S \sim N^\delta$ ($\delta < 1$). The distribution of the degrees of the peaks approaches the degree distribution of the entire network. The distribution of basin sizes now exhibits power-law scaling for small s: $Q(s) \sim s^{-\alpha}$ (or $P(s) \sim s^{-(\alpha+1)}$). We term α the *basin exponent*. The minimal degree m significantly influences the basins count. For $m = 1$ the network is usually fragmented, and thus many basins can form. For $m \geq 2$ the network is connected and consists of a single component, so the number of basins is smaller.

The crossover between the two limiting cases of networks with a giant basin and networks fragmented to many basins is at about $\gamma_c \approx 2.8$. This is revealed by the behavior of the size of the largest basin S: while $S \sim N$ for $\gamma < \gamma_c$, there is no longer a giant basin for $\gamma > \gamma_c$ and $S \sim N^\delta$ with $\delta < 1$ (Fig. 5). The minimum in α also occurs at $\gamma \approx 2.8$ (Fig. 3), and we hypothesize that it is another reflection of the transition.
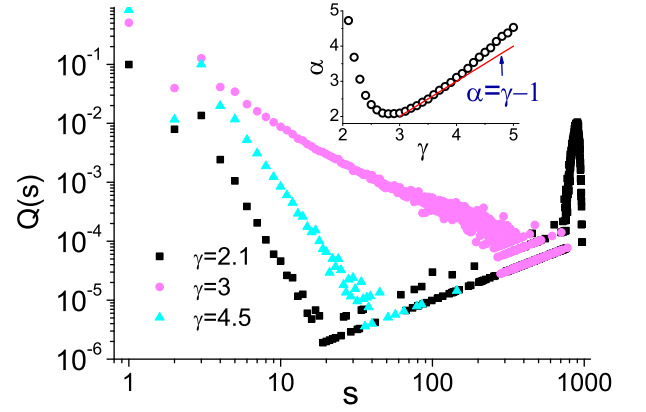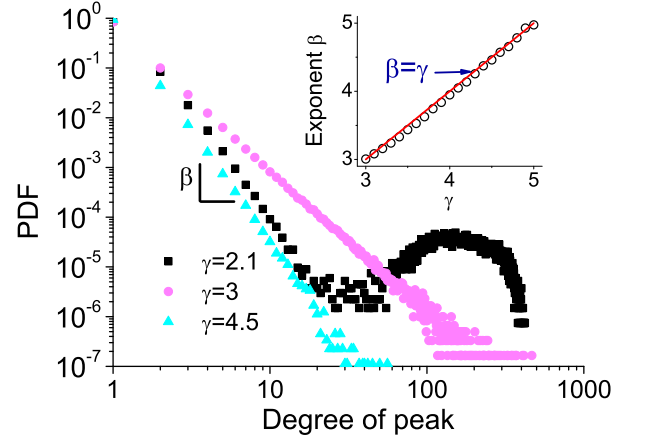
IV. THEORY

A. Random scale-free networks

1. The giant basin

The transition between a network with giant basin to a network fragmented to many basins is observed in the simulations at about $\gamma \approx 2.8$. Interesting questions are whether this transition becomes sharp for infinite systems, and what is the value of γ_c for $N \to \infty$.

A simple argument suggests that for infinite networks a sharp transition occurs at $\gamma_c = 3$. To understand that, consider first the probability of a given node i of degree
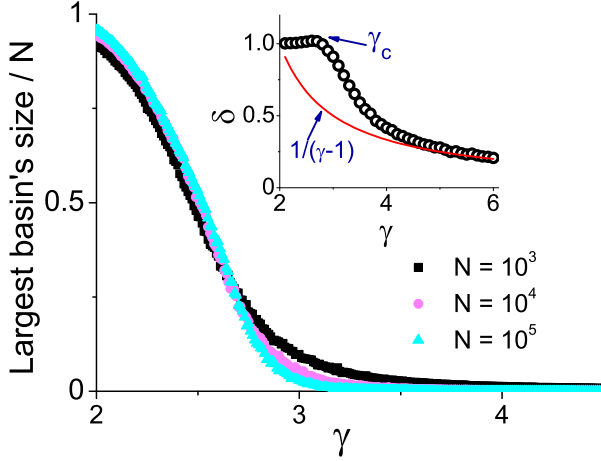
FIG. 5: Size of the largest basin. Plotted is the relative size of the largest basin (S/N) vs. γ for SF networks with $N = 10^3$, 10^4, and 10^5, and $m = 1$. Inset: The largest basin's exponent δ ($S \sim N^\delta$). For $\gamma \lesssim \gamma_c$, $\delta \simeq 1$ ($S \sim N$ is a giant basin), while for large γ, δ approaches $1/(\gamma - 1)$.

k to be a peak,

$$\Pr\{i \text{ is a peak}|k_i = k\} = \left[\sum_{k'=m}^{k} \frac{k'\mathcal{P}(k')}{\langle k \rangle} \right]^k, \quad (1)$$

where $\mathcal{P}(k)$ is the degree distribution, $\langle k \rangle$ is the average degree, and $\frac{k'\mathcal{P}(k')}{\langle k \rangle}$ is the probability that a neighboring node (which is a node followed by a random link) has degree k' [32]. Equation (1) results from the requirement that none of i's k neighbors have degree higher than k. For large k, we substitute $\mathcal{P}(k) \simeq A\,k^{-\gamma}$ and approximate the sum as an integral

$$\sum_{k'=m}^{k} k'\mathcal{P}(k') = \langle k \rangle - \sum_{k'=k+1}^{\infty} k'\mathcal{P}(k')$$
$$\simeq \langle k \rangle - \frac{A}{\gamma - 2}k^{2-\gamma}$$

For $k \gg 1$, Eq. (1) becomes ($\gamma > 2$)

$$\Pr\{i \text{ is a peak}|k_i = k\} = \left[1 - \frac{B}{k^{\gamma-2}}\right]^k \approx \exp\left(-Bk^{3-\gamma}\right),$$

where $B = A/[\langle k \rangle\,(\gamma - 2)]$. Thus, for $\gamma < 3$ the probability of a node to be a peak is small, and approaches zero for large k. Therefore, only the node with the largest degree in the network can be a peak, and it will attract the giant basin. For $\gamma > 3$, every node with large degree is almost surely a peak. For even larger γ, $\gamma > \gamma^* > 3$ (where γ^* is determined by the small k properties of $\mathcal{P}(k)$) there is no longer a giant component in the network. In that case, the size of the largest component scales as $N^{1/(\gamma-1)}$ [33]. The maximal degree of the network has the same scaling. Since the size of largest basin is at least the maximal

degree, but cannot exceed the size of the largest component, we conclude that for $\gamma > \gamma^* > 3$, $S \sim N^\delta$ with $\delta = 1/(\gamma - 1)$. Simulation results support this scaling, (inset of Fig. 5), but it is not known whether a transition is expected, for infinite network, at γ^*.

Another heuristic argument in favor of the phase transition at $\gamma_c = 3$ is the following. Consider two nodes with degrees k_1 and k_2 close to the maximal degree $K \sim N^{1/(\gamma-1)}$ [34]. The probability that these nodes are connected is proportional to $k_1 k_2/N$ [35]. Thus, the probability of the two hubs to be connected scales as $N^{(3-\gamma)/(\gamma-1)}$. Hence, for $\gamma < 3$ the two largest hubs are almost surely connected. The hub with the larger degree attracts the smaller hub, together with its entire basin, to form the giant basin. These arguments are supported by simulation results for increasing values of N (Fig. 5).

2. Number of basins and basin sizes

While we could not obtain a complete derivation of $P(s)$ or $Q(s)$ for random static scale-free networks, it is possible to obtain analytic results for a few chief quantities. Below we derive an exact expression for $\langle n_b \rangle$ as well as reasonable approximations for $\mathrm{Var}(n_b)$ and $Q(1)$.

Clearly, the number of peaks is equal to the number of basins. Thus, the average basin concentration $\langle n_b \rangle = \langle N_b \rangle / N$ is equal to the probability of a node to be a peak. The probability of a given node i with degree k to be a peak is given in Eq. (1). If the degree of i is not specified, we must condition over all possible degrees. Thus,

$$\langle n_b \rangle = \sum_{k=m}^{\infty} \mathcal{P}(k) \left[\sum_{k'=m}^{k} \frac{k'\mathcal{P}(k')}{\langle k \rangle} \right]^k \quad (2)$$

Plugging the degree distribution into (2) completes the derivation; e.g., for SF networks we substitute $\mathcal{P}(k) = k^{-\gamma}/\sum_{k'=m}^{\infty} k'^{-\gamma}$. A comparison of Eq. (2) with simulations yields a perfect agreement (Fig. 2).

Many real-life networks [36], and in particular growing ones, have γ close to 2 and accordingly, a logarithmically diverging average degree $\langle k \rangle \sim \ln N$. Consequently, the $k = m$ term dominates Eq. (2):

$$\langle n_b \rangle \sim \mathcal{P}(m) \left[m\mathcal{P}(m) \right]^m (\ln N)^{-m} + \mathcal{O}[(\ln N)^{-(m+1)}], \quad (3)$$

and we expect $\langle n_b \rangle \sim [\log N]^{-m} \to 0$ for $N \to \infty$ (such that the finite value of $\langle n_b \rangle$ at $\gamma \to 2$ for $m = 1$ in Fig. 2 is a finite size effect).

The calculation of the variance of N_b is more involved since the joint probabilities for multiple peaks are not independent. An approximate expression is given in Appendix B and is plotted in Fig. 10. Rather than the full distribution $Q(s)$, we focus on solitary basins (of size $s = 1$), which account for the bulk of basins. In Appendix B we derive an approximation for $Q(1)$ which is extremely close to simulation results (see Fig. 2).

B. Hierarchical networks

Deterministic hierarchical scale-free networks provide a unique opportunity for an analytical treatment of networks with broad degree distribution [28, 29]. In the following we derive analytical results for the basins topology, which reproduce to some extent the results for random SF networks. In particular, hierarchical networks have a giant basin for small γ, and a power-law distribution of the basin sizes $P(s)$ for large γ, just as was found in Section III for the random networks.

Hierarchical scale-free networks [28, 29] are constructed in a recursive fashion: in (u,v)-flowers, each link in generation n is replaced by two parallel paths consisting of u and v links, to yield generation $n+1$ (Fig. 6); and in (u,v)-trees, defined in analogy to the flowers, we obtain generation $n+1$ of a (u,v)-tree by replacing every link in generation n with a chain of u links, and attaching to each of its endpoints chains of $v/2$ links (assuming v is even). A natural choice for the genus of flowers in generation $n=1$ is a cycle graph (a ring) consisting of $u+v \equiv w$ links and nodes. (u,v)-flowers and trees were shown to have degree distribution of the form $\mathcal{P}(k) \sim k^{-\gamma}$, with $\gamma = 1 + \frac{\ln w}{\ln 2}$, and are thus scale-free. Considering shortest paths, (u,v)-nets with $u=1$ are small-worlds and are otherwise fractals [28, 29]. These and other topological properties, such as clustering and degree-degree correlations make them suitable models for real-life complex networks [1, 2, 37, 38, 39].

We have derived the complete distribution of $P(s)$ for all (u,v)-flowers and trees. This is a tedious exercise in real-space renormalization (verified numerically on a computer) that adds little physical insight. We thus limit the discussion to the results themselves.

For $(1,2)$ and $(1,3)$-flowers, and $(1,2)$-trees, all nodes are evenly split between the w basins peaked in the nodes forming the $n=1$ generation. Thus, this case corresponds to the small γ limit of random SF networks, where a giant basin attracts all nodes.

In $(1,v)$-flowers with $v \geq 4$ (which corresponds to $\gamma > 3$), basins of size $b_m = \frac{2}{3}4^m + \frac{1}{3}$ ($m = 0,1,...,n-2$) appear $(v-3)w^{n-m-1}$ times. Thus, basins of size $s \sim 4^m$ occur with frequency $s^{-\ln w/\ln 4}$. Because the possible basin sizes are not continuous but are exponentially spaced, this leads to a power-law distribution $P(s) \sim s^{-(\alpha+1)}$, with basin exponent $\alpha = \frac{\ln w}{\ln 4} = (\gamma-1)/2$.

In $(1,v)$-trees the situation is qualitatively similar, but more subtle, with different results for $v=4$ and $v>4$. For $(1,4)$-trees, we find that basins of size b_m ($m=2,3,...,n-1$) appear $2w^{n-m-1}$ times. Here $b_m \sim Ar^m$ ($m \gg 1$) where r is the larger root of $r^2 - 5r + 2 = 0$, or $r = \frac{5+\sqrt{17}}{2} = 4.56$. Thus, $P(s) \sim s^{-(\alpha+1)}$ with $\alpha = \frac{\log 5}{\log r} = 1.06$. For $(1,v)$-trees with $v>4$, basins of size $b_m = \frac{5}{24}4^m + \frac{2}{3}$ ($m=2,3,...,n-1$) appear $2w^{n-m}$ times, so $P(s) \sim s^{-(\alpha+1)}$ with $\alpha = \frac{\ln w}{\ln 4}$, as in $(1,v)$-flowers. The size of the largest basin for all $(1,v)$-nets (except $(1,4)$-trees) is $S \sim N^\delta$ with $\delta = 2/(\gamma-1)$.
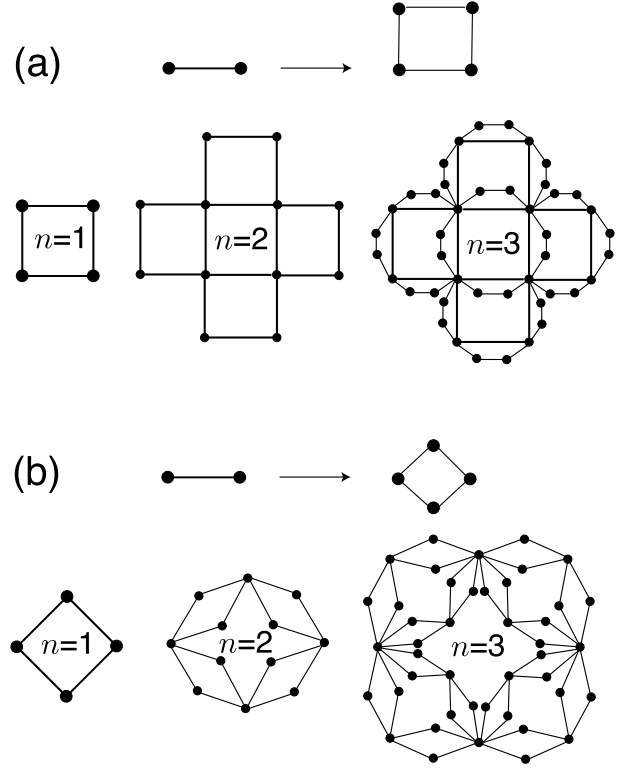

FIG. 6: Hierarcical scale-free (u,v)-flowers. Shown are two examples of networks with degree exponent $\gamma = \ln(u+v)/\ln 2 = 3$, with (a) $u=1$, $v=3$, and (b) $u=2$, $v=2$. In both cases the top of the figure illustrates the edge replacement scheme, to two parallel paths of u and v edges, while the bottom of the figure shows flowers obtained in this way to generations $n = 1$, 2, and 3.

In (u,v)-nets with $u \geq 2$ and $v \geq 2$ the number of basins of size $2^m + 1$ ($1.5 \cdot 2^m + 1$ for $(u,4)$-trees), $m = 3,4,...,n-1$, is $(w-2)w^{n-m}$ for flowers and $[w - 3 + 2/w]w^{n-m}$ for trees. This simply leads to $P(s) \sim s^{-\gamma}$, or $\alpha = \gamma - 1$. Essentially, due to the strong disassortative nature of (u,v)-nets with $u \geq 2$ and $v \geq 2$, the basins in these networks typically consist of a peak and its immediate neighbors, so the basins sizes mirror the degree distribution. Indeed, the size of the largest basin $S \sim N^{\frac{1}{\gamma-1}}$, has the same scaling as the largest degree [28, 34].

To summarize, with the exception of $(1,4)$-trees:

$$\alpha_{(1,v)\text{-nets}} = \begin{cases} (\gamma-1)/2 & \gamma > 3, \\ w \text{ giant basins} & \gamma \leq 3. \end{cases} \quad (4)$$

$$\alpha_{(2,v)\text{-nets}} = \gamma - 1, \quad \gamma \geq 3. \quad (5)$$

Key features revealed by this analysis compare favorably with the results in random scale-free networks. The giant basins found for hierarchical nets with $\gamma \leq 3$ parallels the low-γ phase found in random nets. The power-law decay found for $\gamma > 3$ agrees with the findings for large γ in random nets, as does the increase of α with increasing γ.

V. RANDOM SURFACES

The decomposition of a network into degree-based basins of attraction is a special case of a general problem of finding the basins when the attractiveness of a node is determined by a certain attribute. The association of a scalar field with the network nodes and the emergence of a "gradient network" were suggested in [18, 19] and discussed in Section I. Here, our main interest is in the basins of attractions induced by the external field. In particular, determining the attractiveness of a node by an external parameter allows the basins of attraction to be defined in *regular* networks or lattices where all sites have the same degree. As a basic example, we discuss one- and two-dimensional lattices where each node is assigned a random *height* (or potential energy, density, etc.) The understanding of the topology of such random surfaces is of much importance [40, 41]. For example, the number of peaks determines the number of possible non-satisfied bonds in a spin glass [42, 43] or the "roof" of the surface in ballistic growth models [44].

The height h_i of lattice site i is taken from some distribution (independently of the other lattice sites). Without loss of generality, one may assume the distribution is uniform, in the interval $[0, 1]$. Nodes are attracted to their *shortest* neighbor, so that the surface is energy-like (Fig. 7). The topology of the basins, in this case, has a clear physical interpretation: Put a particle in each node of the lattice and let the particles follow paths of steepest descent. When the system stops evolving, the number of particles s in each minimum is the size of its basin of attraction.
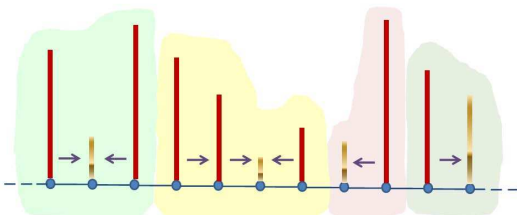

FIG. 7: (Color online) Schematic representation of a random surface in one dimension. Each node is attracted to its shortest neighbor (with periodic boundary conditions). The four highlighted nodes are valleys — their neighbors are taller. All nodes attracted to a valley belong to its basin of attraction. The four basins in the drawing are shown in different background colors.

In one dimension, each point on the surface is either a local maximum (peak), a local minimum (valley), or it has one taller and one shorter neighbor. To find the density of peaks we look at any three consecutive heights (h_1, h_2, h_3) and notice that the probability that h_2 is maximal is $1/3$. Similarly, the density of valleys is also $1/3$. The variance in the number of peaks/valleys can be

derived following similar steps as for networks (Appendix B) and turns out to be $2N/45$ [42].

Let us calculate the probability of a node to be a valley of a basin of size s, $R(s) = \langle n_b \rangle P(s)$. The minimal size $s = 1$ of the basin is obtained in the situation when the minimum is surrounded by two taller heights whose other adjacent heights are shorter than the minimum. If h is the height of the minimum, the above situation occurs with probability $[h(1 - h)]^2$. Integrating over h we find the density of smallest basins

$$R(1) = \int_0^1 dh \, h^2 (1 - h)^2 = \frac{1}{30} \, . \tag{6}$$

For $s \geq 2$, the density of basins of attraction of size s is given by

$$R(s) = \frac{2^{s+3}}{(s+4)!} \, s(s+3) - \frac{4(s^2 + 3s + 1)}{(s+3)!} \, . \tag{7}$$

The derivation of this result is presented in Appendix C. One can verify the validity of both the normalization requirement and the density of valleys:

$$\sum_{s \geq 1} sR(s) = 1 \, , \quad \sum_{s \geq 1} R(s) = \frac{1}{3} = \langle n_b \rangle \, .$$

For large s, $R(s) \sim 1/s!$, which decays much faster than the power-law decay observed for networks.

In two dimensions, basins of attraction are similarly defined as the set of all nodes which are attracted to a given valley. We limit ourselves to the analytical computation of $R(s = 1)$, as larger basins of attraction seem to require very tedious calculations. Let h be the height of the minimum of an $s = 1$-basin of attraction. The adjacent four heights must be taller, which happens with probability $(1 - h)^4$. We write

$$R(1) = \int_0^1 dh \, (1 - h)^4 \sigma(h) \, , \tag{8}$$

and the chief problem is to determine the probability $\sigma(h)$ that for each of the 4 adjacent sites there is a neighbor which is shorter than h. Let $\{x_i\}$ be the heights of diagonal sites $(\pm 1, \pm 1)$, and $\{y_j\}$ the heights of the sites $(0, \pm 2)$ and $(\pm 2, 0)$ [we set the minimum at the origin]. The probability $\sigma(h)$ is given by

$$\begin{aligned} \sigma(h) \; = \; & (1 - h)^4 h^4 + 4h(1 - h)^3 h^2 \\ + \; & 2h^2(1 - h)^2 + 4h^2(1 - h)^2 h \\ + \; & 4h^3(1 - h) + h^4 \, . \end{aligned}$$

Indeed, one possibility is that all the x_i exceed h, and then all the y_j must be shorter than h. This happens with probability $(1 - h)^4 h^4$. If exactly three of the x_i are taller than h, there should be exactly two y_j that are shorter than h. This explains the term $4h(1 - h)^3 h^2$. For the case that two of the x_i are taller and two shorter than h, consideration of their exact locations leads to the

term $2h^2(1-h)^2 + 4h^2(1-h)^2h$. Finally, when at most one x_i is taller than h, there is no requirement on the y_j. Performing the integral in (8) we obtain

$$R(1) = \frac{109}{4290}. \tag{9}$$

In the infinite dimensional case, the random surface is defined on top of a network, as in gradient networks [18, 19]. The only quantity that seems easily calculable is the average number of basins $\langle n_b \rangle$: the probability of a node of degree k to be a valley, for randomly distributed heights, is simply $1/(k+1)$. Thus, for a network,

$$\langle n_b \rangle = \sum_k \mathcal{P}(k)/(k+1). \tag{10}$$

VI. SUMMARY AND DISCUSSION

In summary, we have introduced a process of steepest ascent that partitions complex networks into basins of attraction — subsets of nodes that are attracted to the same peak, the node of highest degree in the basin. For random scale-free networks we find a transition between networks dominated by a giant basin comprising the majority of the nodes, for $\gamma \lesssim \gamma_c$, to numerous, fragmented basins, for $\gamma \gtrsim \gamma_c$. We find numerically that $\gamma_c \approx 2.8$, while theoretical arguments indicate that for $N \to \infty$, $\gamma_c = 3$. Both above and below the transition point, the distribution of finite basins has a power-law tail $s^{-(\alpha+1)}$, where α, the basin exponent, exhibits a non-trivial dependence upon the degree exponent γ. An exact analysis of deterministic hierarchical scale-free nets exhibits some of these features.

A comprehensive description of the complete distribution of basins sizes for static random scale-free networks remains a challenge. Furthermore, other types of networks might exhibit a different basin topology. In particular, randomly growing networks [1, 45], Erdős-Rényi networks [46], and networks with correlations (for example, degree-degree correlations) are of interest and are left for future study.

In a sense, associating each node with a hub and the identification of basins of attraction provides a partition of the network into communities. Numerous algorithms have been proposed to address the problem of classifying nodes into communities. Interestingly, different algorithms employ highly diverse methods and transformations, or measures, of the network topology. For example, many algorithms maximize the modularity index [9] by a wide spectrum of optimization techniques [10, 11, 13, 15]. Others exploit quantities such as betweeness centrality [5, 9], traces of random walk [5, 6, 9, 17], eigenvectors of the network Laplacian [10, 14, 15], electrical conductance [7], and others. While some algorithms recursively split the network into communities separated by "weak links" [5, 6, 7, 11, 13, 15], others take the bottom-top approach and recursively merge highly similar communities, based
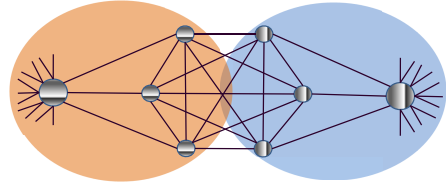

FIG. 8: Illustration of the difference between basins of attraction and network communities. In the plotted toy network, the six nodes in the middle are fully connected and clearly form a single community. However, the three nodes on the left side are attracted to the left hub, whereas the three nodes on the right side are attracted to the right hub. Thus, they are split between two different basins of attraction (indicated with different background colors and different orientation of node fill patterns).

on various similarity indices [9, 10]. Also, while many algorithms output a dendogram (a tree) with partition of the network into disjoint communities at all possible levels of resolution, other studies provide an overlapping community structure; for example, based on identification of almost complete subgraphs [12] or mapping to magnetic domains [8].

How is the partition into basins of attraction compared to other community detectors? First, most algorithms are *global*, since they utilize as much information as possible about the network topology to improve the identification of the communities. In contrast, few other methods (e.g., [12, 47, 48, 49]), including our basins of attraction, are computed in a *local* manner— each node is assigned to a community based only on its immediate neighborhood. Second, and more important, the goal of most community detectors is to find a partition that maximizes intra-community proximity and inter-community separation. That usually takes the form of maximizing the number of links within a community while minimizing the number of links between communities. As opposed to that, our partition to basins of attraction addresses a different question: which nodes are affiliated with the same hub? While in many cases this attribute is correlated with community structure, this is not necessarily always the case, as we demonstrate in Figure 8.

A possible outcome of our analysis is revealed when we test two real-life networks for which the problem of basins is of practical importance: The Internet at the Autonomous Systems (AS) level [50] as of 2007, and the energy landscape's network of Lennard-Jones clusters [25]. Both networks are scale-free, with $\gamma = 2.5, 2.9$, respectively. In both networks there is a giant basin which attracts most nodes (with Verizon's AS being the peak in the Internet), and a few tiny basins, in agreement with the theoretical results for the model scale-free networks. In the Lennard-Jones network, an uphill walk in the degree space, which can be mapped in general onto a downhill walk in the energy landscape, will end up at the node of highest degree, which can be interpreted as the ideal glass state [26]. Note the different situation for the energy landscape of proteins, where the energy increases

with the degree, such that the system is expected to follow a downhill walk in the degree space [20].

For the Internet, the existence of a giant basin implies that a routing scheme that forwards all messages in a steepest ascent manner will quickly arrive at the hub. From the hub, messages could be routed to their target according to a predefined target-specific sequence embedded in the packet, as was previously suggested [23, 51]. This leads to an efficient routing scheme which requires practically no knowledge of the network topology at the nodes, and is thus highly scalable. An obvious drawback of such scheme is the congestion generated at the hubs, which is eliminated in other methods (for example, by routing through shortest paths when the hubs are avoided [52], or by walking down the congestion gradient [22]) Therefore, the steepest ascent search might not be of immediate applicability to the Internet itself, but is however of interest in other newly designed communication networks where the hubs can carry high load. In this context, we note the interesting fact that Boguñá *et. al.* [53] also find a transition between navigable and non-navigable network topology at $\gamma \approx 2.6$, although in their case the navigation is based on minimizing distances within a hidden metric space.

Our partitioning has another potential practical applications for locating the node of highest degree in various search scenarios. A local search starting from a single node and following a steepest ascent would always be successful in networks with a single basin of attraction, as in scale-free networks with $\gamma < \gamma_c$. With more than one basin, a strategy could be devised for starting from a number of randomly selected nodes to find the highest degree with a prescribed rate of success.

A concrete example for such an application is routing in wireless sensor networks [54]. A wireless sensor network is a system consisting of spatially distributed autonomous devices using sensors to cooperatively monitor physical or environmental conditions. In a typical sensor network, one distinguished node serves as a gateway between the sensors and the end users, and must collect data from the nodes. Since energy is usually a very scarce resource at the nodes, an efficient protocol must be designed to transmit the measured data to the base station. Thus, our steepest ascent protocol, in which each node sends out data to its neighbor with highest degree, is of interest. This protocol is expected to be relevant in heterogeneous sensor networks, in which the communication range varies between the nodes [55]. Indeed, we found (data not shown) that for a power-law distribution of communication ranges $\Phi(r) \sim r^{-\epsilon}$, there exist a regime in $(\epsilon, \langle r \rangle)$ parameter space for which the network collapses into a single basin, making the steepest ascent protocol highly efficient.

APPENDIX A: ALTERNATIVE DEFINITION OF BASINS

When the weights of the nodes are taken from a discrete distribution, as in the case where the weight is the degree of the node, neighboring nodes may have the same weight. A method is then required to break the tie. In Section II we presented an algorithm that overcomes this difficulty, which we term the *local search* algorithm. The following *recursive search* algorithm works as well.

Suppose the search starts at node i, and let $j_{\max}$ be the neighbor(s) of i of highest degree $k_{j_{\max}}$. Denote the number of neighbors with degree $k_{j_{\max}}$ as q.

1. If $k_i > k_{j_{\max}}$, i is a peak.

2. If $k_i < k_{j_{\max}}$, i is attracted to $j_{\max}$. (If there is more than one neighbor with degree $k_{j_{\max}}$ (i.e., $q > 1$), select one randomly.)

3. If $k_i = k_{j_{\max}}$, mark i as *visited* and look for the attractor of $j_{\max}$, recursively, among unvisited nodes. If $q > 1$, look also for the attractors of all other neighbors of i with degree $k_{j_{\max}}$. Keep only the attractor of highest degree among the q attractors.

4. If the degree of the attractor of $j_{\max}$ is larger than k_i, i is attracted to $j_{\max}$. If the degree of the attractor of $j_{\max}$ equals k_i, i is a peak.

In other words, in a search for a peak strictly higher than its neighbors, we are allowed to surf over "ridges" of connected nodes of equal degree, until either reaching a peak or a dead end.

Despite the broad distribution of degrees in SF networks, the majority of the nodes have the minimal degree m, or a degree close to m. Thus, one may expect many ridges to form and as a result, a different basin count, depending on whether the local or recursive search is employed. For example, in the hierarchical networks studied in Section 6, a recursive search yields a single giant basin for *all* $(1, v)$-nets. In random SF networks with large γ the recursive search method also yields fewer basins (Fig. 9(a)), which is explained by the prevalence of ridges, in this case, due to the high density of small-degree nodes. However, broader properties of the basins topology remain unaffected by the search
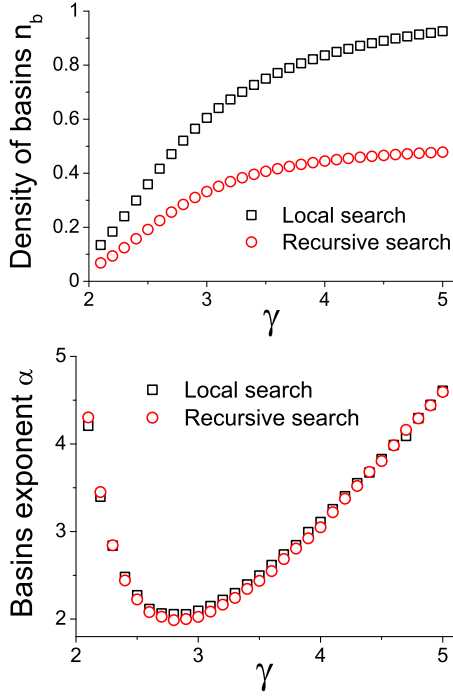
FIG. 9: Basin topology with local vs. recursive search. (a) Density of basins, and (b) the basin exponent α as a function of γ, in SF networks with $N = 1000$ and $m = 1$.

algorithm: $Q(s)$ is practically the same, for large s, as is also the basin exponent α, extracted from either method (Fig. 9(b)).

APPENDIX B: PROPERTIES OF THE NUMBER OF BASINS IN SF NETWORKS

In this appendix, we calculate two quantities related to the number of basins.

1. Variance of the number of basins

Denote by A_i the indicator of the event that node i is a peak, such that $N_b = \sum_{i=1}^{N} A_i$. To compute the variance, we shall use the general formula

$$\text{Var}(N_b) = \sum_{i=1}^{N} \text{Var}(A_i) + 2 \sum_{i=1}^{N} \sum_{j>i}^{N} \text{Cov}(A_i, A_j) \quad \text{(B1)}$$

The first term on the right-hand side is easy to compute:

$$\sum_{i=1}^{N} \text{Var}(A_i) = \sum_{i=1}^{N} P\{A_i\}(1 - P\{A_i\}) = N \langle n_b \rangle (1 - \langle n_b \rangle)$$

Here $P\{A_i\}$ is the probability for A_i to occur. For the second term, we get

$$\begin{aligned} \text{Cov}(A_i, A_j) &= \langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle \\ &= P\{A_i A_j\} - P\{A_i\} P\{A_j\} \end{aligned}$$

Let nodes i and j have degrees k_1 and k_2, respectively. What is the probability for both nodes i and j to be peaks? We condition this probability on whether i and j are connected, which is $k_1 k_2 / (N \langle k \rangle)$ [35]. If they are connected, and they have different degrees, clearly only one of them can serve as a peak, so only the case when both have the same degree k contributes to the covariance. Also, we have to take into account that i and j might share common neighbors. Thus, the probability of i to be a peak is enhanced if j is known to be one. We make the approximation that the number of common neighbors c is fixed once k_i and k_j are given, and is given by:

$$c_{k_i, k_j} \approx (N-2) \sum_{k_\ell=m}^{\infty} \mathcal{P}(k_\ell) \frac{k_i k_\ell}{N \langle k \rangle} \frac{k_j k_\ell}{N \langle k \rangle} \approx \frac{k_i k_j \langle k^2 \rangle}{N \langle k \rangle^2},$$

since this is the probability, summing over all possible degrees of the node $\ell \neq i, j$, that it is adjacent to both i and j. For both i and j to be peaks, if $k_i < k_j$, k_i nodes need to have degree less than k_i, but only $k_j - c$ nodes need to have degree less than k_j (since the c common nodes are guaranteed to have degree less than $k_i < k_j$), and vice-versa if $k_i > k_j$. Approximating the probabilities for two nodes without common neighbors to be peaks as independent, we get,

$$\begin{aligned} \text{Cov}(A_i, A_j) = &\sum_{k=m}^{\infty} k^2 [\mathcal{P}(k)]^2 [f(k)]^{2(k-1)-c_{k,k}} / (N \langle k \rangle) \\ &+ 2 \sum_{k_1=m}^{\infty} \sum_{k_2>k_1} \mathcal{P}(k_1) \mathcal{P}(k_2) \left(1 - \frac{k_1 k_2}{N \langle k \rangle}\right) [f(k_1)]^{k_1} [f(k_2)]^{k_2 - c_{k_1, k_2}} \\ &+ \sum_{k=m}^{\infty} [\mathcal{P}(k)]^2 \left(1 - \frac{k^2}{N \langle k \rangle}\right) [f(k)]^{2k - c_{k,k}} \\ &- \left[\sum_{k=m}^{\infty} \mathcal{P}(k)[f(k)]^k\right]^2, \end{aligned} \quad \text{(B2)}$$

where $f(k) = \sum_{k'=m}^{k} \frac{k' \mathcal{P}(k')}{\langle k \rangle}$ is the probability for a neighbor to have degree no larger than k. The first term corresponds to the case where the nodes are directly connected and have identical degree; the second term is the case when they are not directly connected, and have different degrees; in the third term they are not directly connected but have equal degree; and the last term is just $P\{A_i\} P\{A_j\} = P\{A_i\}^2$. This formula is compared to simulations in Fig. 10 to find a qualitative agreement. We also plot the $\langle n_b \rangle (1 - \langle n_b \rangle)$ term alone, neglecting the covariance, and find that it is a good approximation for the case of large γ.

2. Density of basins of size one

Consider a given node i with degree k and take one of its neighbors j; suppose this neighbor j has degree k'.
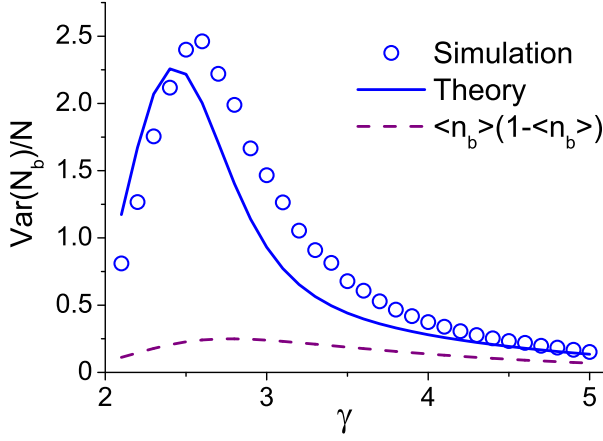
FIG. 10: Variance in number of basins for SF networks with $N = 1000$ and $m = 1$. Symbols represent simulation results, and the solid line corresponds to Eq. (B2). The dashed line is obtained by neglecting the covariance term and has the form $\langle n_b \rangle (1 - \langle n_b \rangle)$ (where $\langle n_b \rangle$ is calculated from Eq. (2)).

For i to be a peak, k' must be less than or equal to k. For i to form a basin of size one, j must have at least one neighbor of degree $k+1$ or above, in order to be attracted to that neighbor and not to i. If we assume that at least up to the second shell, i is a root of a tree, we have:

$$Q(1) = \sum_{k=m}^{\infty} \mathcal{P}(k) \left[\frac{k\mathcal{P}(k)}{\langle k \rangle} + \sum_{k'=m}^{k-1} \frac{k'\mathcal{P}(k')}{\langle k \rangle} q(k, k') \right]^k ,$$

where

$$q(k, k') \equiv 1 - \left[\sum_{k''=m}^{k} \frac{k''\mathcal{P}(k'')}{\langle k \rangle} \right]^{k'-1}$$

is the probability that at least one of the $k'-1$ neighbors (others than i) of j has degree above k. We wrote a separate term for the case of $k' = k$, since in this case we are guaranteed that j is not attracted to i, regardless of the degrees of the neighbors of j. The small correction due to the case when j has another neighbor (other than i) of degree exactly k can be calculated analytically as well, but was found to be negligible. For $\gamma \to 2$, when $\langle k \rangle \sim \log N$, only the $k = m$ term is significant, and thus $Q(1) \sim \mathcal{P}(m) [m\mathcal{P}(m)]^m (\ln N)^{-m}$, and almost all basins are solitary (see Eq. (3)). This is confirmed in the simulations (Fig. 2).

APPENDIX C: BASINS OF ATTRACTION IN ONE DIMENSION

We look at the distribution of basins of attraction in one-dimensional lattices. Consider first a valley separated by distance i from the peak on the left and distance j from the peak on the right, such that particles from both peaks belong to its basin of attraction. The probability of this is

$$R_{ij}^{++} = \int_0^1 dh\, \Pi_i^+(h) \Pi_j^+(h) , \qquad (C1)$$

where, e.g., $\Pi_j^+(h)$ is the probability that j heights to the right of the valley of height h are ascending and the last height is the peak which belongs to the basin of attraction of our valley. The probability $\Pi_k^+(h)$ admits an integral representation

$$\Pi_k^+(h) = \int_{\substack{h<x_1<...<x_k<1 \\ x_{k-1}<x_{k+1}<x_k}} \prod_{a=1}^{k+1} dx_a .$$

Integrating over $x_1, \dots, x_{k-1}$ we recast the above integral into

$$\Pi_k^+(h) = \int_{h<x_{k+1}<x_k<1} dx_k\, dx_{k+1} \frac{(x_{k+1} - h)^{k-1}}{(k-1)!} ,$$

and the remaining integration is trivial:

$$\Pi_k^+(h) = \frac{(1-h)^{k+1}}{(k+1)!} . \qquad (C2)$$

Inserting this equation into (C1) we obtain

$$R_{ij}^{++} = \frac{1}{(i+1)!(j+1)!} \frac{1}{i+j+3} . \qquad (C3)$$

Similarly, we compute

$$R_{ij}^{+-} = \int_0^1 dh\, \Pi_i^+(h) \Pi_j^-(h) , \qquad (C4)$$

where $\Pi_j^-(h)$ is the probability that $j+1$ heights to the right of the valley of height h are ascending and the last height is the peak which belongs to the basin of attraction of the next valley (to its right). The probability $\Pi_k^-(h)$ can be written as

$$\Pi_k^-(h) = \int_{\substack{h<x_1<...<x_{k+1}<1 \\ x_{k+1}>x_{k+2}<x_k}} \prod_{a=1}^{k+2} dx_a .$$

The two last integrations are easily performed,

$$\Pi_k^-(h) = \int_{h<x_1<...<x_k<1} x_k(1-x_k) \prod_{a=1}^{k} dx_a .$$

Integrating over $x_1, \dots, x_{k-1}$ we recast the above integral into

$$\Pi_k^-(h) = \int_h^1 dx_k\, x_k(1-x_k) \frac{(x_k - h)^{k-1}}{(k-1)!} ,$$

which is then computed to yield

$$\Pi_k^-(h) = \frac{(1-h)^{k+1}}{(k+1)!} \left[1 - (1-h) \frac{2}{k+2} \right] . \qquad (C5)$$

Plugging (C2) and (C5) into (C4) we obtain

$$
\begin{aligned}
R_{ij}^{+-} &= \frac{1}{(i+1)!(j+1)!}\frac{1}{i+j+3} \\
&- \frac{1}{(i+1)!(j+2)!}\frac{2}{i+j+4}\,.
\end{aligned} \tag{C6}
$$

Since $R_{ij}^{-+} = R_{ji}^{+-}$, the last quantity to compute is

$$
R_{ij}^{--} = \int_0^1 dh\, \Pi_i^-(h)\Pi_j^-(h)\,. \tag{C7}
$$

Using (C5) we get

$$
\begin{aligned}
R_{ij}^{--} &= \frac{1}{(i+1)!(j+1)!}\frac{1}{i+j+3} \\
&- \frac{1}{(i+1)!(j+1)!}\frac{2}{i+j+4}\left[\frac{1}{i+2}+\frac{1}{j+2}\right] \\
&+ \frac{1}{(i+2)!(j+2)!}\frac{4}{i+j+5}\,.
\end{aligned} \tag{C8}
$$

Equation (C3) is valid when $i \geq 1, j \geq 1$, and the size of the basin of attraction is $s = i + j + 1 \geq 3$. Overall, the density of basins of attraction of type $++$ of size s is

$$
R^{++}(s) = \sum_{\substack{i\geq 1, j\geq 1 \\ i+j=s-1}} R_{ij}^{++} = \frac{2^{s+1}-2}{(s+2)!} - \frac{2}{(s+2)s!}\,. \tag{C9}
$$

Equation (C6) is valid when $i \geq 1, j \geq 0$. The density of basins of attraction of type $+-$ of size s is

$$
R^{+-}(s) = \sum_{\substack{i\geq 1, j\geq 0 \\ i+j=s-1}} R_{ij}^{+-}\,.
$$

Computing the sum we find

$$
\begin{aligned}
R^{+-}(s) &= \frac{2^{s+1}-2}{(s+2)!} - \frac{1}{(s+2)s!} \\
&- 2\frac{2^{s+2}-2}{(s+3)!} + \frac{4}{(s+3)(s+1)!}\,.
\end{aligned} \tag{C10}
$$

Of course, $R^{+-}(s) = R^{-+}(s)$, and (C10) is valid when $s \geq 2$.

Equation (C8) is valid when $i \geq 0, j \geq 0$. The density of basins of attraction of size s and type $--$ is given by

$$
R^{--}(s) = \sum_{\substack{i\geq 0, j\geq 0 \\ i+j=s-1}} R_{ij}^{--}\,.
$$

Computing the sum we find

$$
\begin{aligned}
R^{--}(s) &= \frac{2^{s+1}-2}{(s+2)!} - 4\frac{2^{s+2}-2}{(s+3)!} + \frac{4}{(s+3)(s+1)!} \\
&+ 4\frac{2^{s+3}-2}{(s+4)!} - \frac{8}{(s+4)(s+2)!}\,,
\end{aligned} \tag{C11}
$$

which is valid for $s \geq 1$. Defining $R^{++}(1) = R^{++}(2) = 0$ and $R^{+-}(1) = R^{-+}(1) = 0$, we finally have ($s \geq 1$):

$$
R(s) = R^{++}(s) + R^{+-}(s) + R^{-+}(s) + R^{--}(s)\,. \tag{C12}
$$

Inserting (C9),(C10), and (C11) into (C12) we arrive at the announced result (7).

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